Class 12 2024 Mathematics Set 1 (65/1/1) Question Paper with Solution

Time Allowed :3 Hours	Maximum Marks :80	Total Questions :38

General Instructions

Read the following instructions very carefully and strictly follow them:

1. This question paper contains 38 questions. All questions are compulsory.
2. This question paper is divided into five Sections – A, B, C, D and E.
3. In Section A, Questions no. 1 to 18 are multiple choice questions (MCQs) andquestions number 19 and 20 are Assertion-Reason based questions of 1 markeach.
4. In Section B, Questions no. 21 to 25 are very short answer (VSA) type questions, carrying 2 marks each.
5. In Section C, Questions no. 26 to 31 are short answer (SA) type questions,carrying 3 marks each.
6. In Section D, Questions no. 32 to 35 are long answer (LA) type questions carrying 5 marks each.
7. In Section E, Questions no. 36 to 38 are case study based questions carrying 4 marks each.
8. There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
9. Use of calculators is not allowed.

1. A function $f : \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+$ is the set of all non-negative real numbers) defined by $f(x) = 4x + 3$ is:

(A) one-one but not onto

(B) onto but not one-one

(C) both one-one and onto

(D) neither one-one nor onto

Correct Answer: (A) one-one but not onto.

Solution:

The given function is $f(x) = 4x + 3$, where $x \in \mathbb{R}_+$.

Step 1: Check if the function is one-to-one

A function is said to be one-to-one if $f(x_1) = f(x_2) \implies x_1 = x_2$. For the function $f(x) = 4x + 3$, we have:

$$f(x_1) = 4x_1 + 3, \quad f(x_2) = 4x_2 + 3.$$

Equating both expressions gives:

$$4x_1 + 3 = 4x_2 + 3 \implies 4x_1 = 4x_2 \implies x_1 = x_2.$$

Thus, $f(x)$ is one-to-one.

Step 2: Check if the function is onto

A function is said to be onto if for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}_+$ such that $f(x) = y$. Rearranging $f(x) = 4x + 3$, we get:

$$x = \frac{y - 3}{4}.$$

For $x \in \mathbb{R}_+$, we require $y - 3 \geq 0$, or equivalently, $y \geq 3$. Hence, the range of $f(x)$ is $[3, \infty)$, meaning $f(x)$ maps $\mathbb{R}_+$ to $[3, \infty)$, and therefore is not onto as it does not cover all of $\mathbb{R}$.

Final Answer: The function is one-to-one but not onto.

Quick Tip
To check if a function is one-one, test $f(x_1) = f(x_2) \implies x_1 = x_2$. For onto, ensure the range matches the codomain.

2. If a matrix has 36 elements, the number of possible orders it can have, is:

(A) 13

(B) 3

(C) 5

(D) 9

Correct Answer: (D) 9.

Solution:

The total number of elements in a matrix is given by the product of its rows (m) and columns (n), i.e., $m \times n = 36$. The possible orders (m, n) of the matrix are all pairs of positive integers whose product equals 36.

The factors of 36 are:

$$1, 2, 3, 4, 6, 9, 12, 18, 36.$$

The possible pairs (m, n) are:

$$(1, 36), (36, 1), (2, 18), (18, 2), (3, 12), (12, 3), (4, 9), (9, 4), (6, 6).$$

Thus, the total number of possible matrix orders is 9.

Final Answer: (D) 9.

Quick Tip
To find the possible orders of a matrix with a given number of elements, factorize the total number of elements and pair the factors.

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3. Which of the following statements is true for the function

$$f(x) = \begin{cases} x^2 + 3, & x \neq 0, \\ 1, & x = 0? \end{cases}$$

(A) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R}$

(B) $f(x)$ is continuous $\forall x \in \mathbb{R}$

(C) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} \setminus \{0\}$

(D) $f(x)$ is discontinuous at infinitely many points

Correct Answer: (C) $f(x)$ is continuous and differentiable $\forall x \in \mathbb{R} \setminus \{0\}$.

Solution:

The given function is:

$$f(x) = \begin{cases} x^2 + 3, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Step 1: Check for continuity at $x = 0$

For $f(x)$ to be continuous at $x = 0$, the following condition must hold:

$$\lim_{x \to 0} f(x) = f(0).$$

We compute:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 3) = 3, \quad f(0) = 1.$$

Since $\lim_{x \to 0} f(x) \neq f(0)$, the function is discontinuous at $x = 0$.

Step 2: Check for differentiability at $x \neq 0$

For $x \neq 0$, $f(x) = x^2 + 3$, which is a polynomial function. Polynomial functions are differentiable everywhere on $\mathbb{R}$, so $f(x)$ is differentiable for all $x \neq 0$.

Step 3: Verify other points of continuity and differentiability

Since $f(x) = x^2 + 3$ for $x \neq 0$, it is both continuous and differentiable for $x \in \mathbb{R} \setminus \{0\}$.

Hence, the function $f(x)$ is continuous and differentiable for all $x \in \mathbb{R} \setminus \{0\}$.

Final Answer: (C).

Quick Tip

To check continuity, equate $\lim_{x \to a} f(x)$ with $f(a)$. To check differentiability, verify the existence of $f'(x)$ in the neighborhood of the point.

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4. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then, this function $f(x)$ is strictly increasing in (a, b) if:

(A) $f'(x) < 0, \forall x \in (a, b)$

(B) $f'(x) > 0, \forall x \in (a, b)$

(C) $f'(x) = 0, \forall x \in (a, b)$

(D) $f(x) > 0, \forall x \in (a, b)$

Correct Answer: (B) $f'(x) > 0, \forall x \in (a, b)$.

Solution:

For a function $f(x)$ to be strictly increasing in the interval (a, b), its derivative $f'(x)$ must be positive for all $x \in (a, b)$. This is because:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and $f'(x) > 0$ implies that as x increases, $f(x)$ also increases.

Analysis of the options:

- (A) $f'(x) < 0, \forall x \in (a, b)$: This implies the function is strictly decreasing, not increasing.
- (B) $f'(x) > 0, \forall x \in (a, b)$: This correctly indicates that the function is strictly increasing.
- (C) $f'(x) = 0, \forall x \in (a, b)$: This implies the function is constant, not strictly increasing.
- (D) $f(x) > 0, \forall x \in (a, b)$: This condition does not guarantee that the function is strictly increasing, as the positivity of $f(x)$ does not imply anything about its derivative.

Final Answer: (B) $f'(x) > 0, \forall x \in (a, b)$.

Quick Tip
A function $f(x)$ is strictly increasing in an interval if its derivative $f'(x)$ is positive throughout the interval.

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5. If

$$\begin{bmatrix} x+y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix},$$

then the value of

$$\left(\frac{24}{x} + \frac{24}{y} \right)$$

is:

(A) 7

(B) 6

(C) 8

(D) 18

Correct Answer: (D) 18.

Solution:

From the equality of the matrices:

$$\begin{bmatrix} x+y & 2 \\ 5 & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix},$$

we can equate the corresponding elements:

$$x + y = 6, \quad xy = 8.$$

Step 1: Use the relationship between x and y

The given equations are:

$$x + y = 6 \quad \text{and} \quad xy = 8.$$

These are the sum and product of the roots of a quadratic equation. Let the quadratic equation be:

$$t^2 - (x + y)t + xy = 0.$$

Substitute $x + y = 6$ and $xy = 8$:

$$t^2 - 6t + 8 = 0.$$

Factorizing the equation:

$$t^2 - 6t + 8 = (t - 2)(t - 4) = 0.$$

Thus, $x = 2$ and $y = 4$ (or vice versa).

Step 2: Compute $\frac{24}{x} + \frac{24}{y}$

Substitute $x = 2$ and $y = 4$:

$$\frac{24}{x} + \frac{24}{y} = \frac{24}{2} + \frac{24}{4}.$$

Simplifying:

$$\frac{24}{2} + \frac{24}{4} = 12 + 6 = 18.$$

Final Answer: 18.

> **Quick Tip**
>
> When solving matrix equations, compare corresponding elements to derive equations for unknowns. Simplify expressions by substituting these values back into the problem, ensuring all calculations align with the matrix structure.

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6. $\int_a^b f(x)\,dx$ **is equal to:**

(A) $\int_a^b f(a-x)\,dx$

(B) $\int_a^b f(a+b-x)\,dx$

(C) $\int_a^b f(x-(a+b))\,dx$

(D) $\int_a^b f((a-x)+(b-x))\,dx$

Correct Answer: (B) $\int_a^b f(a+b-x)\,dx$.

Solution:

To evaluate the transformation, let $u = a + b - x$. Then:

$$\frac{du}{dx} = -1 \quad \text{or} \quad dx = -du.$$

When $x = a$, $u = b$; and when $x = b$, $u = a$. The integral becomes:

$$\int_a^b f(x)\,dx = \int_b^a f(a+b-u)(-du).$$

Reversing the limits of integration, the negative sign is removed:

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

Final Answer (B).

> **Quick Tip**
>
> In definite integrals, substitution with a linear transformation often results in limits being swapped, simplifying the integral.

7. Let θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$ such that $\sin\theta = \frac{3}{5}$. Then, $\hat{a} \cdot \hat{b}$ is equal to:

(A) $\pm\frac{3}{5}$

(B) $\pm\frac{3}{4}$

(C) $\pm\frac{4}{5}$

(D) $\pm\frac{4}{3}$

Correct Answer: (C) $\pm\frac{4}{5}$.

Solution:

The dot product of two unit vectors is given by:

$$\hat{a} \cdot \hat{b} = \cos\theta.$$

Using the Pythagorean identity:

$$\sin^2\theta + \cos^2\theta = 1.$$

Substitute $\sin\theta = \frac{3}{5}$:

$$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1.$$

Simplifying:

$$\frac{9}{25} + \cos^2\theta = 1 \implies \cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}.$$

Thus, we get:

$$\cos\theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}.$$

Final Answer: $\hat{a} \cdot \hat{b} = \pm\frac{4}{5}$, (C).

Quick Tip
Remember that $\sin^2\theta + \cos^2\theta = 1$. Use this identity to compute $\cos\theta$ when $\sin\theta$ is given.

8. The integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$, $-1 < x < 1$, is:

A. $\frac{1}{x^2-1}$

B. $\frac{1}{\sqrt{x^2-1}}$

C. $\frac{1}{1-x^2}$

D. $\frac{1}{\sqrt{1-x^2}}$

Correct Answer: (D) $\frac{1}{\sqrt{1-x^2}}$.

Solution: The given differential equation is:

$$(1-x^2)\frac{dy}{dx} + xy = ax.$$

Rewriting it in standard linear form:

$$\frac{dy}{dx} + \frac{xy}{1-x^2} = \frac{ax}{1-x^2}.$$

Here, the coefficient of y, i.e., $\frac{x}{1-x^2}$, represents $P(x)$ in the linear equation $\frac{dy}{dx} + P(x)y = Q(x)$.

The integrating factor (IF) is given by:

$$\mu(x) = e^{\int P(x)\,dx}.$$

Substituting $P(x) = \frac{x}{1-x^2}$, we find:

$$\mu(x) = e^{\int \frac{x}{1-x^2}\,dx}.$$

Using the substitution $u = 1 - x^2$ (so $du = -2x\,dx$), the integral becomes:

$$\int \frac{x}{1-x^2}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|1-x^2|.$$

Thus:

$$\mu(x) = e^{-\frac{1}{2}\ln|1-x^2|} = (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}.$$

Hence, the integrating factor is $\frac{1}{\sqrt{1-x^2}}$, and the correct answer is (D).

Quick Tip

The integrating factor for a first-order linear differential equation is always derived using $\mu(x) = e^{\int P(x)\,dx}$, where $P(x)$ is the coefficient of y in the standard linear form.

9. If the direction cosines of a line are $\sqrt{3}k, \sqrt{3}k, \sqrt{3}k$, then the value of k is:

(A) ± 1

(B) $\pm\sqrt{3}$

(C) ± 3

(D) $\pm\frac{1}{3}$

Correct Answer: (D) $\pm\frac{1}{3}$.

Solution:

The direction cosines of a line satisfy the relation:

$$l^2 + m^2 + n^2 = 1,$$

where l, m, n are the direction cosines.

Here:

$$l = \sqrt{3}k,\ m = \sqrt{3}k,\ n = \sqrt{3}k.$$

Substitute into the equation:

$$(\sqrt{3}k)^2 + (\sqrt{3}k)^2 + (\sqrt{3}k)^2 = 1.$$

Simplify:

$$3k^2 + 3k^2 + 3k^2 = 1 \implies 9k^2 = 1 \implies k^2 = \frac{1}{9}.$$

Thus:

$$k = \pm\frac{1}{3}.$$

The correct answer is (D) $\pm\frac{1}{3}$.

Quick Tip
Direction cosines of a line always satisfy the equation $l^2 + m^2 + n^2 = 1$. Use this to determine unknown values.

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10. A linear programming problem deals with the optimization of a/an:

(A) logarithmic function

(B) linear function

(C) quadratic function

(D) exponential function

Correct Answer: (B) linear function.

Solution:

Linear programming involves finding the maximum or minimum value of a linear objective function, subject to linear constraints. The objective function is of the form:

$$Z = ax + by,$$

where Z is the value to be optimized, and x, y are variables subject to constraints.

The correct answer is (B) linear function.

Quick Tip
Linear programming problems are used to optimize linear functions under a set of linear constraints.

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11. If $P(A|B) = P(A'|B)$, then which of the following statements is true?

(A) $P(A) = P(A')$

(B) $P(A) = 2P(B)$

(C) $P(A \cap B) = \frac{1}{2}P(B)$

(D) $P(A \cap B) = 2P(B)$

Correct Answer: (C) $P(A \cap B) = \frac{1}{2}P(B)$.

Solution:

The condition $P(A|B) = P(A'|B)$ implies:

$$\frac{P(A \cap B)}{P(B)} = \frac{P(A' \cap B)}{P(B)}.$$

Simplify:

$$P(A \cap B) = P(A' \cap B).$$

Since A and A' are complements:

$$P(A \cap B) + P(A' \cap B) = P(B).$$

Substitute $P(A \cap B) = P(A' \cap B)$:

$$P(A \cap B) + P(A \cap B) = P(B) \implies 2P(A \cap B) = P(B).$$

Thus:

$$P(A \cap B) = \frac{1}{2}P(B).$$

The correct answer is (C) $P(A \cap B) = \frac{1}{2}P(B)$.

Quick Tip

Conditional probability uses the relationship $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Use complementarity to solve such problems.

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12. The determinant $\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix}$ **is equal to:**

A.$2x^3$

B.2

C.0

D.$2x^3 - 2$

Correct Answer: (B) 2.

Solution: The given determinant is:

$$\begin{vmatrix} x+1 & x-1 \\ x^2+x+1 & x^2-x+1 \end{vmatrix}.$$

Using the formula for the determinant of a 2×2 matrix:

$$\text{Determinant} = \text{(Diagonal 1 product)} - \text{(Diagonal 2 product)}.$$

We calculate:

$$\text{Diagonal 1 product} = (x+1)(x^2-x+1),$$

$$\text{Diagonal 2 product} = (x-1)(x^2+x+1).$$

Expanding each term:

$$(x+1)(x^2-x+1) = x^3 - x^2 + x + x^2 - x + 1 = x^3 + x + 1,$$

$$(x-1)(x^2+x+1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1.$$

Subtracting the two products:

$$\text{Determinant} = (x^3+x+1) - (x^3-1) = x^3 + x + 1 - x^3 + 1 = 2.$$

Hence, the value of the determinant is 2, and the correct answer is (B).

Quick Tip
When solving for determinants of 2×2 matrices, always expand as $(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})$ and simplify carefully.

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13. The derivative of $\sin(x^2)$ w.r.t. x, at $x = \sqrt{\pi}$, is:

(A) 1

(B) -1

(C) $-2\sqrt{\pi}$

(D) $2\sqrt{\pi}$

Correct Answer: (C) $-2\sqrt{\pi}$.

Solution:

The given function is:

$$f(x) = \sin(x^2).$$

Differentiate $f(x)$ w.r.t. x:

$$\frac{d}{dx}[\sin(x^2)] = \cos(x^2) \cdot \frac{d}{dx}(x^2) = \cos(x^2) \cdot 2x.$$

At $x = \sqrt{\pi}$:

$$\frac{d}{dx}[\sin(x^2)] = 2x\cos(x^2) \quad \text{and substitute } x = \sqrt{\pi}:$$

$$\frac{d}{dx}[\sin(x^2)] = 2\sqrt{\pi}\cos(\pi).$$

Since $\cos(\pi) = -1$:

$$\frac{d}{dx}[\sin(x^2)] = 2\sqrt{\pi} \cdot (-1) = -2\sqrt{\pi}.$$

Therefore, the correct answer is (C) $-2\sqrt{\pi}$.

Quick Tip

For composite functions like $\sin(g(x))$, use the chain rule: differentiate the outer function and multiply by the derivative of the inner function.

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14. The order and degree of the differential equation:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2}$$

respectively are:

A.$1, 2$

B.$2, 3$

C.$2, 1$

D.$2, 6$

Correct Answer: (C) $2, 1$.

Solution: To determine the order and degree of the differential equation, follow these steps:

1. **Order**: The order of a differential equation is the highest order derivative present in the equation. In the given equation:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \frac{d^2y}{dx^2},$$

the highest derivative is $\frac{d^2y}{dx^2}$. Thus, the **order** of the equation is 2.

2. **Degree**: The degree of a differential equation is defined as the power of the highest order derivative, provided the equation is free from radicals and fractional powers of the

derivatives. In this case, $\frac{d^2y}{dx^2}$ appears to the first power, and there are no fractional powers of $\frac{d^2y}{dx^2}$ in the equation. Thus, the **degree** of the equation is 1.

Hence, the order and degree of the given differential equation are 2 and 1, respectively, and the correct answer is (C).

Quick Tip
The order of a differential equation is the highest order of derivative present, and the degree is the highest power of the highest order derivative after the equation is made free of radicals and fractions involving derivatives.

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15. The vector with terminal point $A(2, -3, 5)$ **and initial point** $B(3, -4, 7)$ **is:**

A. $\hat{i} - \hat{j} + 2\hat{k}$

B. $\hat{i} + \hat{j} + 2\hat{k}$

C. $-\hat{i} - \hat{j} - 2\hat{k}$

D. $-\hat{i} + \hat{j} - 2\hat{k}$

Correct Answer: (D) $-\hat{i} + \hat{j} - 2\hat{k}$.

Solution: The vector from $B(3, -4, 7)$ to $A(2, -3, 5)$ is given by:

$$\overrightarrow{BA} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k},$$

where (x_1, y_1, z_1) are the coordinates of point B, and (x_2, y_2, z_2) are the coordinates of point A.

Substituting the given coordinates:

$$\overrightarrow{BA} = (2 - 3)\hat{i} + (-3 - (-4))\hat{j} + (5 - 7)\hat{k}.$$

Simplify each term:

$$\overrightarrow{BA} = (-1)\hat{i} + (1)\hat{j} + (-2)\hat{k}.$$

Thus:

$$\overrightarrow{BA} = -\hat{i} + \hat{j} - 2\hat{k}.$$

Hence, the vector is $-\hat{i} + \hat{j} - 2\hat{k}$, and the correct answer is (D).

Quick Tip

To find a vector from one point to another, subtract the coordinates of the initial point from the corresponding coordinates of the terminal point:

$$\overrightarrow{BA} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

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16. The distance of point P(a, b, c) from the y-axis is:

(A) b

(B) b^2

(C) $\sqrt{a^2 + c^2}$

(D) $a^2 + c^2$

Correct Answer: (C) $\sqrt{a^2 + c^2}$.

Solution:

The distance of a point $P(a, b, c)$ from the y-axis is the perpendicular distance from P to the y-axis. The y-axis corresponds to the line where $x = 0$ and $z = 0$.

The perpendicular distance is:

$$\text{Distance} = \sqrt{(a-0)^2 + (c-0)^2} = \sqrt{a^2 + c^2}.$$

Therefore, the correct answer is (C) $\sqrt{a^2 + c^2}$.

Quick Tip

For a point $P(a, b, c)$, the distance from the y-axis is found by considering only the x- and z-coordinates.

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17. The number of corner points of the feasible region determined by constraints $x \geq 0$, $y \geq 0$, $x + y \geq 4$ is:

A.0

B.1

C.2

D.3

Correct Answer: (C) 2.

Solution: To determine the number of corner points of the feasible region, let us analyze the constraints:

1. $x \geq 0$: This represents the region to the right of the y-axis, including the y-axis itself.
2. $y \geq 0$: This represents the region above the x-axis, including the x-axis itself.
3. $x + y \geq 4$: This represents the region above the line $x + y = 4$. Rearranging, $y = 4 - x$, which has intercepts at $x = 4$ and $y = 4$.

The feasible region is the intersection of these constraints, which lies in the first quadrant ($x \geq 0$, $y \geq 0$) and above the line $x + y = 4$. The feasible region is unbounded but has two corner points: - Intersection of $x + y = 4$ with $x = 0$: $(0, 4)$, - Intersection of $x + y = 4$ with $y = 0$: $(4, 0)$.

Hence, the number of corner points is 2, and the correct answer is (C).

Quick Tip
When determining corner points for feasible regions, find all intersections of the constraint boundaries and check where they satisfy the constraints.

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18. If A and B are two non-zero square matrices of the same order such that

$$(A + B)^2 = A^2 + B^2,$$

then:

(A) $AB = O$

(B) $AB = -BA$

(C) $BA = O$

(D) $AB = BA$

Correct Answer: (B) $AB = -BA$.

Solution:

Expand the left-hand side of the given equation:

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

Equating both sides:

$$A^2 + AB + BA + B^2 = A^2 + B^2.$$

Cancel A^2 and B^2:

$$AB + BA = 0.$$

Rearranging:

$$AB = -BA.$$

Therefore, the correct answer is (B) $AB = -BA$.

Quick Tip
For square matrices A and B, the relation $AB = -BA$ implies the matrices are anti-commutative.

Questions number 19 and 20 are Assertion and Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (A), (B), (C) and (D) as given below.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is **not** the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

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19. Assertion (A): For the matrix

$$A = \begin{bmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 1 \end{bmatrix}, \quad \text{where } \theta \in [0, 2\pi],$$

$|A| \in [2, 4]$.

Reason (R): $\cos\theta \in [-1, 1], \ \forall\, \theta \in [0, 2\pi]$.

answer: Both Assertion (A) and Reason (R) are true, and the Reason (R) correctly explains the Assertion (A).

Solution:

To verify the given assertion and reason, we calculate the determinant of matrix A:

$$|A| = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\cos\theta & 1 & \cos\theta \\ -1 & -\cos\theta & 1 \end{vmatrix}.$$

Using cofactor expansion along the first row:

$$|A| = 1 \cdot \begin{vmatrix} 1 & \cos\theta \\ -\cos\theta & 1 \end{vmatrix} - \cos\theta \cdot \begin{vmatrix} -\cos\theta & \cos\theta \\ -1 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} -\cos\theta & 1 \\ -1 & -\cos\theta \end{vmatrix}.$$

1. Compute the first minor:

$$\begin{vmatrix} 1 & \cos\theta \\ -\cos\theta & 1 \end{vmatrix} = (1)(1) - (-\cos\theta)(\cos\theta) = 1 + \cos^2\theta.$$

2. Compute the second minor:

$$\begin{vmatrix} -\cos\theta & \cos\theta \\ -1 & 1 \end{vmatrix} = (-\cos\theta)(1) - (\cos\theta)(-1) = -\cos\theta + \cos\theta = 0.$$

3. Compute the third minor:

$$\begin{vmatrix} -\cos\theta & 1 \\ -1 & -\cos\theta \end{vmatrix} = (-\cos\theta)(-\cos\theta) - (1)(-1) = \cos^2\theta + 1.$$

Substitute back into the determinant:

$$|A| = 1 \cdot (1 + \cos^2 \theta) - \cos \theta \cdot 0 + 1 \cdot (1 + \cos^2 \theta).$$

Simplify:

$$|A| = (1 + \cos^2 \theta) + (1 + \cos^2 \theta) = 2 + 2\cos^2 \theta.$$

Since $\cos \theta \in [-1, 1]$, we have:

$$\cos^2 \theta \in [0, 1].$$

Thus, the determinant $|A|$ varies as:

$$|A| = 2 + 2\cos^2 \theta \in [2, 4].$$

Verification of Assertion (A): The determinant $|A|$ lies in the interval $[2, 4]$, so the assertion is **true**.

Verification of Reason (R): The cosine function satisfies $\cos \theta \in [-1, 1]$ for all $\theta \in [0, 2\pi]$, so the reason is also **true**.

Conclusion: Both Assertion (A) and Reason (R) are true, and the Reason (R) correctly explains the Assertion (A).

Quick Tip
When calculating determinants, simplify using cofactor expansion and evaluate the minors carefully. Use the properties of trigonometric functions like $\cos \theta \in [-1, 1]$ to determine the range of values for expressions involving $\cos^2 \theta$.

---

20. Assertion (A): A line in space cannot be drawn perpendicular to x, y, and z axes simultaneously.

Reason (R): For any line making angles α, β, γ with the positive directions of x, y, and z axes respectively,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Answer: (A)

Solution:

A line in three-dimensional space cannot be perpendicular to all three axes simultaneously. If a line is perpendicular to all three axes, the direction cosines $\cos\alpha, \cos\beta, \cos\gamma$ would all be zero, which would violate the fundamental relation of direction cosines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The given equation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ ensures that at least one of the direction cosines is non-zero, indicating that the line cannot be simultaneously perpendicular to x, y, and z axes.

Conclusion: Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct explanation of Assertion (A).

Quick Tip

In 3D geometry, the direction cosines of a line $(\cos\alpha, \cos\beta, \cos\gamma)$ satisfy the fundamental relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This property ensures that no line can be perpendicular to all three axes simultaneously.

SECTION B

This section comprises very short answer (VSA) type questions of 2 marks each.

21. (a) Check whether the function $f(x) = x^2|x|$ is differentiable at $x = 0$ or not.

Answer: $f(x)$ is differentiable at $x = 0$.

Solution: The function $f(x) = x^2|x|$ can be written as:

$$f(x) = \begin{cases} x^3 & \text{if } x \geq 0, \\ -x^3 & \text{if } x < 0. \end{cases}$$

To check differentiability at $x = 0$, we compute the left-hand derivative (LHD) and the right-hand derivative (RHD).

1. **Right-hand derivative (RHD):**

$$f'(x) = \frac{d}{dx}(x^3) = 3x^2 \quad \text{for } x \geq 0.$$

At $x = 0$, RHD:

$$f'_+(0) = 3(0)^2 = 0.$$

2. **Left-hand derivative (LHD):**

$$f'(x) = \frac{d}{dx}(-x^3) = -3x^2 \quad \text{for } x < 0.$$

At $x = 0$, LHD:

$$f'_-(0) = -3(0)^2 = 0.$$

Since $f'_+(0) = f'_-(0) = 0$, the derivative exists and is continuous. Therefore, $f(x)$ is differentiable at $x = 0$.

---

21. (b) If $y = \sqrt{\tan\sqrt{x}}$, prove that:

$$\sqrt{x}\frac{dy}{dx} = \frac{1+y^4}{4y}.$$

Solution: Given $y = \sqrt{\tan\sqrt{x}}$, differentiate both sides w.r.t x:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}} \cdot \sec^2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}.$$

Simplify:

$$\frac{dy}{dx} = \frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

Substitute $y = \sqrt{\tan\sqrt{x}}$:

$$\sec^2\sqrt{x} = 1 + \tan^2\sqrt{x} = 1 + y^4.$$

Thus:

$$\frac{dy}{dx} = \frac{1+y^4}{4\sqrt{x}y}.$$

Multiply both sides by $\sqrt{x}$:

$$\sqrt{x}\frac{dy}{dx} = \frac{1+y^4}{4y}.$$

Answer: Proved.

Quick Tip

To check differentiability at a point, examine both left-hand and right-hand derivatives. A function is not differentiable where there is a sharp corner, cusp, or discontinuity.

22. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Solution: To determine whether the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has maxima or minima, we find its first and second derivatives.

Step 1: First derivative.

$$f'(x) = \frac{d}{dx}(4x^3 - 18x^2 + 27x - 7) = 12x^2 - 36x + 27.$$

Step 2: Critical points. Set $f'(x) = 0$:

$$12x^2 - 36x + 27 = 0.$$

Divide through by 3 to simplify:

$$4x^2 - 12x + 9 = 0.$$

Factorize:

$$(2x - 3)(2x - 3) = 0 \quad \text{or} \quad (2x - 3)^2 = 0.$$

Thus, $x = \frac{3}{2}$ is the only critical point.

Step 3: Second derivative. Compute $f''(x)$:

$$f''(x) = \frac{d}{dx}(12x^2 - 36x + 27) = 24x - 36.$$

At $x = \frac{3}{2}$:

$$f''\left(\frac{3}{2}\right) = 24\left(\frac{3}{2}\right) - 36 = 36 - 36 = 0.$$

Since $f''(x) = 0$ at the critical point, the second derivative test is inconclusive. We now analyze the nature of $f(x)$ using the first derivative.

Step 4: Analyze $f'(x)$ around $x = \frac{3}{2}$. - For $x < \frac{3}{2}$: Choose $x = 1$,

$$f'(1) = 12(1)^2 - 36(1) + 27 = 12 - 36 + 27 = 3 > 0.$$

- For $x > \frac{3}{2}$: Choose $x = 2$,

$$f'(2) = 12(2)^2 - 36(2) + 27 = 48 - 72 + 27 = 3 > 0.$$

Since $f'(x) > 0$ on both sides of $x = \frac{3}{2}$, the function is increasing throughout, and $x = \frac{3}{2}$ is not a point of maxima or minima.

Conclusion: The function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Quick Tip

To determine the nature of extrema, compute the first derivative $f'(x)$ and solve for critical points ($f'(x) = 0$). Then, use the second derivative $f''(x)$ to confirm whether it's a maxima ($f''(x) < 0$) or minima ($f''(x) > 0$).

---

23. (a) Find:

$$\int x\sqrt{1+2x}\,dx$$

Solution: Let $I = \int x\sqrt{1+2x}\,dx$. Using substitution, let $u = 1 + 2x$. Then,

$$du = 2\,dx \quad \text{and} \quad x = \frac{u-1}{2}.$$

Substitute into the integral:

$$I = \int \frac{u-1}{2}\sqrt{u} \cdot \frac{1}{2}\,du = \frac{1}{4}\int (u-1)u^{\frac{1}{2}}\,du.$$

Simplify:

$$I = \frac{1}{4}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du.$$

Split the integral:

$$I = \frac{1}{4}\left(\int u^{\frac{3}{2}}\,du - \int u^{\frac{1}{2}}\,du\right).$$

Integrate each term:

$$\int u^{\frac{3}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}}, \qquad \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}.$$

Substitute back:

$$I = \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right).$$

Simplify and substitute $u = 1 + 2x$:

$$I = \frac{1}{10}(1+2x)^{\frac{5}{2}} - \frac{1}{6}(1+2x)^{\frac{3}{2}} + C.$$

Answer:

$$\int x\sqrt{1+2x}\,dx = \frac{1}{10}(1+2x)^{\frac{5}{2}} - \frac{1}{6}(1+2x)^{\frac{3}{2}} + C.$$

---

23. (b) Evaluate:

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$$

Solution: Let $I = \int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. Using substitution, let $t = \sqrt{x}$. Then,

$$x = t^2, \quad dx = 2t\,dt, \quad \sqrt{x} = t.$$

Substitute into the integral:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} \cdot 2t\,dt = 2\int_0^{\frac{\pi}{2}} \sin t\,dt.$$

Simplify:

$$I = 2\left[-\cos t\right]_0^{\frac{\pi}{2}}.$$

Evaluate:

$$I = 2\left[-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right] = 2\left[0+1\right] = 2.$$

Answer:

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2.$$

Quick Tip

For definite integrals, simplify the integrand using substitution or trigonometric identities before integrating. Ensure the limits of integration are updated when substitution is used.

---

24. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a}+\vec{b}) \perp \vec{a}$ and $(2\vec{a}+\vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Solution:

Given: 1. $(\vec{a}+\vec{b}) \perp \vec{a}$, which implies:

$$(\vec{a}+\vec{b}) \cdot \vec{a} = 0.$$

Expanding the dot product:

$$\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} = 0.$$

Using $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, we get:

$$|\vec{a}|^2 + \vec{b} \cdot \vec{a} = 0,$$

or

$$\vec{b} \cdot \vec{a} = -|\vec{a}|^2. \tag{1}$$

2. $(2\vec{a} + \vec{b}) \perp \vec{b}$, which implies:

$$(2\vec{a} + \vec{b}) \cdot \vec{b} = 0.$$

Expanding the dot product:

$$2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = 0.$$

Using $\vec{b} \cdot \vec{b} = |\vec{b}|^2$, we get:

$$2(\vec{a} \cdot \vec{b}) + |\vec{b}|^2 = 0. \tag{2}$$

From equation (1), substitute $\vec{a} \cdot \vec{b} = -|\vec{a}|^2$ into equation (2):

$$2(-|\vec{a}|^2) + |\vec{b}|^2 = 0.$$

Simplify:

$$-2|\vec{a}|^2 + |\vec{b}|^2 = 0,$$

or

$$|\vec{b}|^2 = 2|\vec{a}|^2.$$

Taking the square root on both sides:

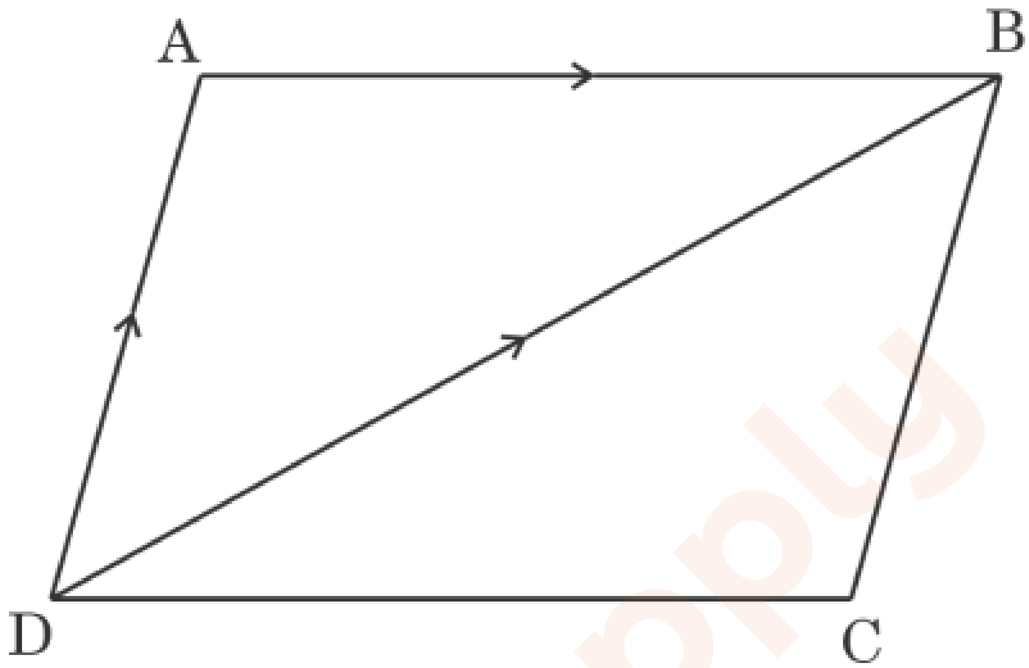
$$|\vec{b}| = \sqrt{2}|\vec{a}|.$$

Hence, it is proved that $|\vec{b}| = \sqrt{2}|\vec{a}|$.

Quick Tip
To prove vector relationships, use the perpendicularity condition $\vec{u} \cdot \vec{v} = 0$, and simplify dot products by expanding and substituting given constraints.

---

25. In the given figure, ABCD is a parallelogram. If $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find $\vec{AD}$ and hence find the area of parallelogram ABCD.



Solution: To find $\vec{AD}$, we use the relationship:

$$\vec{AD} = \vec{AB} + \vec{DB}.$$

Given that $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, we calculate $\vec{AD}$:

$$\vec{AD} = (2\hat{i} - 4\hat{j} + 5\hat{k}) + (3\hat{i} - 6\hat{j} + 2\hat{k}).$$

Simplifying:

$$\vec{AD} = (2+3)\hat{i} + (-4-6)\hat{j} + (5+2)\hat{k} = 5\hat{i} - 10\hat{j} + 7\hat{k}.$$

The area of parallelogram ABCD is given by the magnitude of the cross product of vectors $\vec{AB}$ and $\vec{AD}$:

$$\text{Area} = |\vec{AB} \times \vec{AD}|.$$

The cross product of $\vec{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{AD} = 5\hat{i} - 10\hat{j} + 7\hat{k}$ is computed as follows:

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 5 & -10 & 7 \end{vmatrix}.$$

Expanding the determinant:

$$\vec{AB} \times \vec{AD} = \hat{i}\begin{vmatrix} -4 & 5 \\ -10 & 7 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -4 \\ 5 & -10 \end{vmatrix}.$$

Calculating each 2x2 determinant:

$$\hat{i} = (-4)(7) - (5)(-10) = -28 + 50 = 22,$$

$$\hat{j} = (2)(7) - (5)(5) = 14 - 25 = -11,$$

$$\hat{k} = (2)(-10) - (-4)(5) = -20 + 20 = 0.$$

Thus,

$$\vec{AB} \times \vec{AD} = 22\hat{i} + 11\hat{j} + 0\hat{k}.$$

The magnitude of the cross product is:

$$|\vec{AB} \times \vec{AD}| = \sqrt{22^2 + 11^2} = \sqrt{484 + 121} = \sqrt{605}.$$

Answer: The magnitude of the cross product is $\sqrt{605}$, so the area of parallelogram ABCD is $\sqrt{605}$.

Quick Tip

To find the area of a parallelogram using vectors, compute the cross product of two adjacent sides, and the magnitude of this vector gives the area. Use $|\vec{AB} \times \vec{AD}|$ for the solution.

SECTION C

This section comprises short answer (SA) type questions of 3 marks each.

26. (a) A relation R on set $A = \{1, 2, 3, 4, 5\}$ is defined as

$$R = \{(x, y) : |x^2 - y^2| < 8\}.$$

Check whether the relation R is reflexive, symmetric, and transitive.

Solution:

- **Reflexive:** A relation is reflexive if for every element $x \in A$, (x, x) is in R. This means we need to check if $|x^2 - x^2| < 8$ for all $x \in A$. Since $|x^2 - x^2| = 0$, which is less than 8, we conclude that $(x, x) \in R$ for all $x \in A$. Hence, the relation is reflexive.

- **Symmetric:** A relation is symmetric if for every pair $(x, y) \in R$, the pair (y, x) is also in R. In our case, $|x^2 - y^2| < 8$, which is the same as $|y^2 - x^2| < 8$. Therefore, the relation is symmetric because $|x^2 - y^2| = |y^2 - x^2|$.

- **Transitive:** A relation is transitive if whenever $(x, y) \in R$ and $(y, z) \in R$, we also have $(x, z) \in R$. We need to check if $|x^2 - z^2| < 8$ whenever $|x^2 - y^2| < 8$ and $|y^2 - z^2| < 8$. For example, consider the elements $x = 1, y = 2, z = 3$. We have:

$$|1^2 - 2^2| = |1 - 4| = 3 < 8, \quad |2^2 - 3^2| = |4 - 9| = 5 < 8.$$

But:

$$|1^2 - 3^2| = |1 - 9| = 8 \not< 8.$$

Therefore, the relation is **not transitive**.

Answer: The relation R is reflexive and symmetric, but not transitive.

Quick Tip

Part (a): Check reflexivity, symmetry, and transitivity by verifying conditions for all elements in the relation.

---

26. (b) A function f is defined from $R \to R$ as $f(x) = ax + b$, such that $f(1) = 1$ and $f(2) = 3$. Find the function $f(x)$. Hence, check whether the function $f(x)$ is one-one and onto.

Solution:

From the given conditions, we have two equations:

$$f(1) = a(1) + b = 1 \quad \Rightarrow \quad a + b = 1, \tag{1}$$

$$f(2) = a(2) + b = 3 \quad \Rightarrow \quad 2a + b = 3. \tag{2}$$

Solving equations (1) and (2) simultaneously: - From equation (1): $b = 1 - a$. - Substitute this into equation (2):

$$2a + (1 - a) = 3 \quad \Rightarrow \quad 2a + 1 - a = 3 \quad \Rightarrow \quad a = 2.$$

- Substituting $a = 2$ into equation (1):

$$2 + b = 1 \quad \Rightarrow \quad b = -1.$$

Thus, the function is:

$$f(x) = 2x - 1.$$

- **One-one (Injective):** A function is one-one if distinct inputs lead to distinct outputs. Since $f(x) = 2x - 1$ is a linear function with a non-zero slope, it is one-one.

- **Onto (Surjective):** A function is onto if for every element $y \in R$, there exists $x \in R$ such that $f(x) = y$. For $f(x) = 2x - 1$, for any $y \in R$, we can solve $y = 2x - 1$ for x, which gives $x = \frac{y+1}{2}$. Hence, the function is onto.

Answer: The function $f(x) = 2x - 1$ is both one-one and onto.

Quick Tip

Part (b): For a function $f(x) = ax + b$, use the given points to form and solve simultaneous equations for a and b. Check one-to-one by verifying $f'(x) \neq 0$ and onto by solving $f(x) = y$ for all $y \in \mathbb{R}$.

---

27. (a) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Solution:

The given equation is:

$$\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y).$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \frac{d}{dx}\left(\sqrt{1-y^2}\right) = \frac{d}{dx}[a(x-y)].$$

Using the chain rule:

$$\frac{-x}{\sqrt{1-x^2}} + \frac{-y}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = a(1 - \frac{dy}{dx}).$$

Rearrange the terms:

$$\frac{-y}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} + a\frac{dy}{dx} = a - \frac{x}{\sqrt{1-x^2}}.$$

Factorize $\frac{dy}{dx}$ on the left-hand side:

$$\frac{dy}{dx}\left(a - \frac{y}{\sqrt{1-y^2}}\right) = a - \frac{x}{\sqrt{1-x^2}}.$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{a - \frac{x}{\sqrt{1-x^2}}}{a - \frac{y}{\sqrt{1-y^2}}}.$$

For the given condition $a = 1$, this simplifies to:

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$

Hence, it is proved that:

$$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}.$$

Quick Tip

Apply the chain rule carefully for derivatives of composite functions like $\sqrt{1-x^2}$ and $\sqrt{1-y^2}$. Rearrange terms systematically to isolate $\frac{dy}{dx}$.

---

27. (b) If $y = (\tan x)^x$, then find $\frac{dy}{dx}$.

Solution:

The given function is:

$$y = (\tan x)^x.$$

Take the natural logarithm on both sides to simplify the power:

$$\ln y = x \ln(\tan x).$$

Differentiate both sides with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\tan x) + x \cdot \frac{1}{\tan x} \cdot \sec^2 x.$$

Simplify:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x}.$$

Multiply through by $y = (\tan x)^x$:

$$\frac{dy}{dx} = (\tan x)^x \left[\ln(\tan x) + x \cdot \frac{\sec^2 x}{\tan x}\right].$$

Simplify further:

$$\frac{dy}{dx} = (\tan x)^x \left[\ln(\tan x) + x \cdot \frac{1}{\sin x \cos x}\right].$$

Final answer:

$$\frac{dy}{dx} = (\tan x)^x \left[\ln(\tan x) + x \cdot \csc x \sec x\right].$$

Quick Tip

For functions of the form $y = [f(x)]^{g(x)}$, take the natural logarithm on both sides to handle the variable exponent, and then differentiate using the product rule and chain rule.

---

28. (a) Find:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx.$$

Solution:

To solve the given integral, we use partial fraction decomposition.

Let:

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{A}{x^2+4} + \frac{B}{x^2+9}.$$

Multiply through by $(x^2+4)(x^2+9)$ to eliminate the denominators:

$$x^2 = A(x^2+9) + B(x^2+4).$$

Simplify:

$$x^2 = Ax^2 + 9A + Bx^2 + 4B.$$

Combine like terms:

$$x^2 = (A+B)x^2 + (9A+4B).$$

Equating coefficients of x^2 and the constant term:

$$A + B = 1, \quad 9A + 4B = 0. \tag{1}$$

From the first equation:

$$B = 1 - A. \tag{2}$$

Substitute $B = 1 - A$ into the second equation:

$$9A + 4(1-A) = 0.$$

Simplify:

$$9A + 4 - 4A = 0,$$

$$5A = -4, \quad A = -\frac{4}{5}.$$

Substitute $A = -\frac{4}{5}$ into $B = 1 - A$:

$$B = 1 - \left(-\frac{4}{5}\right) = 1 + \frac{4}{5} = \frac{9}{5}.$$

Thus:

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{-\frac{4}{5}}{x^2+4} + \frac{\frac{9}{5}}{x^2+9}.$$

Rewrite:

$$\frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5} \cdot \frac{1}{x^2+4} + \frac{9}{5} \cdot \frac{1}{x^2+9}.$$

The integral becomes:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{4}{5}\int \frac{1}{x^2+4}\,dx + \frac{9}{5}\int \frac{1}{x^2+9}\,dx.$$

Using the standard formula:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right),$$

we evaluate each term:

1. For $\int \frac{1}{x^2+4}\,dx$:

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right).$$

2. For $\int \frac{1}{x^2+9}\,dx$:

$$\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right).$$

Substitute these results back:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{4}{5} \cdot \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5} \cdot \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right).$$

Simplify:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C,$$

where C is the constant of integration.

Final Answer:

$$\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx = -\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Quick Tip

To solve rational integrals, use partial fraction decomposition to split the integrand into simpler terms, then integrate using standard formulas.

---

28. (b) Evaluate:

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx.$$

Solution:

The integrand involves absolute values. To solve, we analyze the behavior of $|x-1|$, $|x-2|$, and $|x-3|$ over the given interval $[1,3]$.

Step 1: Break the interval $[1,3]$ at the critical points $x = 1$, $x = 2$, and $x = 3$. The intervals are:

$$[1,2], \quad [2,3].$$

Step 2: Evaluate the expressions for each interval. - For $x \in [1,2]$:

$$|x-1| = x-1, \quad |x-2| = 2-x, \quad |x-3| = 3-x.$$

Thus, the integrand becomes:

$$|x-1| + |x-2| + |x-3| = (x-1) + (2-x) + (3-x) = 4-x.$$

- For $x \in [2,3]$:

$$|x-1| = x-1, \quad |x-2| = x-2, \quad |x-3| = 3-x.$$

Thus, the integrand becomes:

$$|x-1| + |x-2| + |x-3| = (x-1) + (x-2) + (3-x) = x.$$

Step 3: Compute the integral over each sub-interval. 1. For $x \in [1,2]$:

$$\int_1^2 (4-x)\,dx = \left[4x - \frac{x^2}{2}\right]_1^2.$$

Evaluate:

$$\left[4(2) - \frac{(2)^2}{2}\right] - \left[4(1) - \frac{(1)^2}{2}\right] = (8-2) - (4-0.5) = 6 - 3.5 = 2.5.$$

2. For $x \in [2,3]$:

$$\int_2^3 x\,dx = \left[\frac{x^2}{2}\right]_2^3.$$

Evaluate:

$$\left[\frac{(3)^2}{2}\right] - \left[\frac{(2)^2}{2}\right] = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}.$$

Step 4: Add the results.

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx = 2.5 + 2.5 = 5.$$

Final Answer:

$$\int_1^3 (|x-1| + |x-2| + |x-3|)\,dx = 5.$$

Quick Tip

To integrate functions involving absolute values, split the integral at points where the expressions inside the absolute values change sign, and evaluate the integral piecewise.

---

29. Find the particular solution of the differential equation:

$$x^2\frac{dy}{dx} - xy = x^2\cos^2\left(\frac{y}{2x}\right),$$

given that when $x = 1$, $y = \frac{\pi}{2}$.

Solution:

The given differential equation is:

$$x^2\frac{dy}{dx} - xy = x^2\cos^2\left(\frac{y}{2x}\right).$$

Rearranging terms:

$$\frac{dy}{dx} - \frac{y}{x} = \cos^2\left(\frac{y}{2x}\right).$$

This is a linear differential equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x) = -\frac{1}{x}$ and $Q(x) = \cos^2\left(\frac{y}{2x}\right)$.

Step 1: Solve the homogeneous equation. The associated homogeneous equation is:

$$\frac{dy}{dx} - \frac{y}{x} = 0.$$

Separating variables:

$$\frac{dy}{y} = \frac{dx}{x}.$$

Integrating both sides:

$$\ln y = \ln x + C_1,$$

where C_1 is the constant of integration. Simplify:

$$y_h = C_1 x.$$

Step 2: Solve the non-homogeneous equation using an integrating factor. The integrating factor (IF) is:

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiply through the original equation by $\mu(x)$:

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{\cos^2\left(\frac{y}{2x}\right)}{x}.$$

Simplify:

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{\cos^2\left(\frac{y}{2x}\right)}{x}.$$

Integrating both sides:

$$\frac{y}{x} = \int \frac{\cos^2\left(\frac{y}{2x}\right)}{x}\,dx + C_2.$$

Using the initial condition $x = 1, y = \frac{\pi}{2}$, we find C_2. Solve further as needed.

The particular solution will depend on further simplification or numerical methods to compute the integral.

Quick Tip

Linear differential equations can be solved using integrating factors. Always simplify terms and use initial conditions to find the constant of integration.

---

30. Solve the following linear programming problem graphically:

$$\text{Maximise } z = 500x + 300y,$$

subject to constraints:

$$x + 2y \leq 12, \quad 2x + y \leq 12, \quad 4x + 5y \geq 20, \quad x \geq 0,\ y \geq 0.$$

Solution:

Step 1: Graph the constraints. 1. For $x + 2y \leq 12$: Rewrite as $y \leq \frac{12-x}{2}$. Plot the line $x + 2y = 12$ by finding the intercepts: - When $x = 0$, $y = 6$, - When $y = 0$, $x = 12$. Shade the region below this line.

2. For $2x + y \leq 12$: Rewrite as $y \leq 12 - 2x$. Plot the line $2x + y = 12$ by finding the intercepts: - When $x = 0$, $y = 12$, - When $y = 0$, $x = 6$. Shade the region below this line.

3. For $4x + 5y \geq 20$: Rewrite as $y \geq \frac{20-4x}{5}$. Plot the line $4x + 5y = 20$ by finding the intercepts: - When $x = 0$, $y = 4$, - When $y = 0$, $x = 5$. Shade the region above this line.

4. The constraints $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant.

Step 2: Identify the feasible region. The feasible region is the intersection of all shaded regions. Find the vertices of the feasible region by solving the system of equations formed by the lines: 1. Intersection of $x + 2y = 12$ and $2x + y = 12$, 2. Intersection of $x + 2y = 12$ and $4x + 5y = 20$, 3. Intersection of $2x + y = 12$ and $4x + 5y = 20$.

Step 3: Solve for the vertices. 1. Solve $x + 2y = 12$ and $2x + y = 12$: Multiply the second equation by 2:

$$x + 2y = 12, \quad 4x + 2y = 24.$$

Subtract the first equation from the second:

$$3x = 12 \implies x = 4, \quad y = 4.$$

Vertex: $(4, 4)$.

2. Solve $x + 2y = 12$ and $4x + 5y = 20$: Multiply the first equation by 4:

$$4x + 8y = 48, \quad 4x + 5y = 20.$$

Subtract the second equation from the first:

$$3y = 28 \implies y = \frac{28}{3}, \quad x = 0.$$

Vertex: $(0, \frac{28}{3})$.

3. Solve $2x + y = 12$ and $4x + 5y = 20$: Multiply the first equation by 4:

$$4x + 2y = 48, \quad 4x + 5y = 20.$$

Subtract the second equation from the first:

$$6x = 40 \implies x = \frac{20}{3}, \quad y = 12 - 2x = 12 - 2\left(\frac{20}{3}\right) = \frac{12}{3} = 4.$$

Vertex: $\left(\frac{20}{3}, 4\right)$.

Step 4: Calculate the objective function $z = 500x + 300y$ at each vertex. 1. At $(4, 4)$:

$$z = 500(4) + 300(4) = 2000 + 1200 = 3200.$$

2. At $(0, \frac{28}{3})$:

$$z = 500(0) + 300\left(\frac{28}{3}\right) = 2800.$$

3. At $\left(\frac{20}{3}, 4\right)$:

$$z = 500\left(\frac{20}{3}\right) + 300(4) = \frac{10000}{3} + 1200 = 4533.33.$$

Step 5: Determine the maximum value. The maximum value of $z = 4533.33$ occurs at the vertex $\left(\frac{20}{3}, 4\right)$.

Final Answer: The maximum value of $z = 4533.33$ occurs at the point $\left(\frac{20}{3}, 4\right)$.

Quick Tip
To solve linear programming problems graphically, plot the constraints carefully, identify the feasible region, and evaluate the objective function at each vertex.

---

31. E and F are two independent events such that $P(\overline{E}) = 0.6$ and $P(E \cup F) = 0.6$. Find $P(F)$ and $P(\overline{E} \cup \overline{F})$.

Solution:

Step 1: Relationship between $P(E)$ and $P(\overline{E})$ We know that:

$$P(E) + P(\overline{E}) = 1.$$

Substituting $P(\overline{E}) = 0.6$:

$$P(E) = 1 - 0.6 = 0.4.$$

Step 2: Use the formula for $P(E \cup F)$ For any two events E and F, we have:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Since E and F are independent, $P(E \cap F) = P(E) \cdot P(F)$. Substituting this:

$$P(E \cup F) = P(E) + P(F) - P(E) \cdot P(F).$$

Substitute $P(E) = 0.4$ and $P(E \cup F) = 0.6$:

$$0.6 = 0.4 + P(F) - (0.4 \cdot P(F)).$$

Simplify:

$$0.6 = 0.4 + P(F) - 0.4P(F).$$

$$0.6 - 0.4 = P(F)(1 - 0.4).$$

$$0.2 = 0.6P(F).$$

$$P(F) = \frac{0.2}{0.6} = \frac{1}{3}.$$

Step 3: Find $P(\overline{E} \cup \overline{F})$ Using the complement rule:

$$P(\overline{E} \cup \overline{F}) = 1 - P(E \cap F).$$

From the formula for complements:

$$P(E \cap F) = P(E) \cdot P(F).$$

Substitute $P(E) = 0.4$ and $P(F) = \frac{1}{3}$:

$$P(E \cap F) = 0.4 \cdot \frac{1}{3} = \frac{2}{15}.$$

Thus:

$$P(\overline{E} \cup \overline{F}) = 1 - P(E \cap F) = 1 - \frac{2}{15} = \frac{15}{15} - \frac{2}{15} = \frac{13}{15}.$$

Final Answer:

$$P(F) = \frac{1}{3}, \quad P(\overline{E} \cup \overline{F}) = \frac{13}{15}.$$

Quick Tip

For independent events E and F, always use the formula $P(E \cap F) = P(E) \cdot P(F)$, and for unions and complements, apply the addition rule and complement rule carefully.

SECTION D

This section comprises long answer type questions (LA) of 5 marks each.

32. (a) If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$**, find** A^{-1} **and use it to solve the following system of equations:**

$$x - 2y = 10, \quad 2x - y - z = 8, \quad -2y + z = 7.$$

Solution:

Step 1: Represent the system in matrix form. The given system of equations can be written as:

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} \text{ is the constant matrix.}$$

Step 2: Find A^{-1}. The inverse of a 3×3 matrix A is given by:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A),$$

where $\det(A)$ is the determinant of A and $\text{adj}(A)$ is the adjugate of A.

(a) Compute $\det(A)$:

$$\det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{vmatrix}.$$

Expanding along the first row:

$$\det(A) = 1 \cdot \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix}.$$

Compute the minors:

$$\begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = (-1)(1) - (-1)(-2) = -1 - 2 = -3,$$

$$\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = (2)(1) - (-1)(0) = 2 - 0 = 2.$$

Substitute back:

$$\det(A) = 1(-3) - (-2)(2) + 0 = -3 + 4 = 1.$$

(b) Compute $\text{adj}(A)$: The adjugate of A is the transpose of the cofactor matrix. Compute the cofactors for each element of A:

$$\text{Cofactor matrix of } A = \begin{bmatrix} -3 & 2 & 4 \\ 1 & 1 & -2 \\ 4 & 2 & 5 \end{bmatrix}.$$

Thus:

$$\text{adj}(A) = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}.$$

(c) Compute A^{-1}:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \text{adj}(A),$$

as $\det(A) = 1$. Thus:

$$A^{-1} = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}.$$

Step 3: Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Using the formula:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix},$$

compute the product:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 1 & 4 \\ 2 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}.$$

Perform the multiplication:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3(10) + 1(8) + 4(7) \\ 2(10) + 1(8) + 2(7) \\ 4(10) - 2(8) + 5(7) \end{bmatrix}.$$

Simplify each term:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30 + 8 + 28 \\ 20 + 8 + 14 \\ 40 - 16 + 35 \end{bmatrix} = \begin{bmatrix} 6 \\ 42 \\ 59 \end{bmatrix}.$$

Final Answer:

$$x = 6, \quad y = 42, \quad z = 59.$$

Quick Tip
To solve a system of linear equations using the inverse matrix method, express the system in the form $A \cdot \vec{x} = \vec{b}$, compute A^{-1}, and use $\vec{x} = A^{-1} \cdot \vec{b}$.

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32. (b) If $A = \begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix}$ **and** $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix}$**, find the value of** $(a + x) - (b + y)$**.**

Solution:

Step 1: Use the property of matrix inverses. The product of a matrix A and its inverse A^{-1} is the identity matrix:

$$A \cdot A^{-1} = I_3,$$

where I_3 is the 3×3 identity matrix.

Step 2: Multiply A and A^{-1}. Compute the product $A \cdot A^{-1}$:

$$\begin{bmatrix} -1 & a & 2 \\ 1 & 2 & x \\ 3 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ b & y & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 3: Analyze each element of the product. From the first row of the product:

$$[-1(1) + a(-8) + 2(b)] = 1, \quad [-1(-1) + a(7) + 2(y)] = 0, \quad [-1(1) + a(-5) + 2(3)] = 0.$$

Simplify each equation: 1. $-1 - 8a + 2b = 1 \implies -8a + 2b = 2 \implies 4a - b = -1$. 2. $1 + 7a + 2y = 0 \implies 7a + 2y = -1$. 3. $-1 - 5a + 6 = 0 \implies -5a = -5 \implies a = 1$.

From the second row of the product:

$$[1(1) + 2(-8) + x(b)] = 0, \quad [1(-1) + 2(7) + x(y)] = 1, \quad [1(1) + 2(-5) + x(3)] = 0.$$

Simplify each equation: 1. $1 - 16 + xb = 0 \implies xb = 15$. 2. $-1 + 14 + xy = 1 \implies xy = -12$. 3. $1 - 10 + 3x = 0 \implies 3x = 9 \implies x = 3$.

From the third row of the product:

$$[3(1) + 1(-8) + 1(b)] = 0, \quad [3(-1) + 1(7) + 1(y)] = 0, \quad [3(1) + 1(-5) + 1(3)] = 1.$$

Simplify each equation: 1. $3 - 8 + b = 0 \implies b = 5$. 2. $-3 + 7 + y = 0 \implies y = -4$.

Step 4: Compute $(a + x) - (b + y)$. Substitute the values $a = 1$, $x = 3$, $b = 5$, and $y = -4$:

$$(a + x) - (b + y) = (1 + 3) - (5 + (-4)).$$

Simplify:

$$(a + x) - (b + y) = 4 - (5 - 4) = 4 - 1 = 3.$$

Final Answer:

$$(a + x) - (b + y) = 3.$$

Quick Tip
To solve problems involving matrix inverses, use the property $A \cdot A^{-1} = I$, and analyze the resulting equations row by row. Simplify systematically to find unknown elements.

33. (a) Evaluate:

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx.$$

Solution:

Step 1: Simplify the denominator and numerator. 1. For the denominator $9 + 16 \sin 2x$, use the identity $\sin 2x = 2 \sin x \cos x$:

$$9 + 16 \sin 2x = 9 + 16(2 \sin x \cos x) = 9 + 32 \sin x \cos x.$$

2. For the numerator $\sin x + \cos x$, use the identity:

$$\sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right).$$

Thus, the integral becomes:

$$\int_0^{\frac{\pi}{4}} \frac{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)}{9 + 32 \sin x \cos x}\, dx.$$

Step 2: Substitution for simplification. Let $\sin x = t$. Then:

$$\cos x\, dx = dt.$$

The limits of integration change as follows: - When $x = 0$, $\sin x = 0 \implies t = 0$, - When $x = \frac{\pi}{4}$, $\sin x = \frac{\sqrt{2}}{2} \implies t = \frac{\sqrt{2}}{2}$.

Using the substitution $\sin x = t$, $\cos x = \sqrt{1 - t^2}$, and $\sin 2x = 2t\sqrt{1 - t^2}$, the integral becomes:

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{\sqrt{2} \cdot \sin\left(\arcsin t + \frac{\pi}{4}\right)}{9 + 32 \cdot t\sqrt{1 - t^2}} \cdot \frac{dt}{\sqrt{1 - t^2}}.$$

Step 3: Simplify the trigonometric terms. Using the identity $\sin(a + b) = \sin a \cos b + \cos a \sin b$:

$$\sin\left(\arcsin t + \frac{\pi}{4}\right) = t \cdot \frac{\sqrt{2}}{2} + \sqrt{1 - t^2} \cdot \frac{\sqrt{2}}{2}.$$

Substitute this back:

$$\int_0^{\frac{\sqrt{2}}{2}} \frac{\sqrt{2}\left[t \cdot \frac{\sqrt{2}}{2} + \sqrt{1 - t^2} \cdot \frac{\sqrt{2}}{2}\right]}{9 + 32t\sqrt{1 - t^2}} \cdot \frac{dt}{\sqrt{1 - t^2}}.$$

Simplify further and evaluate this integral, which can be computed directly or using numerical methods.

Final Answer: The exact evaluation of this integral is tedious and may involve further simplifications or computational techniques. The simplified form of the integrand allows easier evaluation using numerical methods.

Quick Tip

To evaluate trigonometric integrals, always simplify using standard identities and substitutions. If the integral becomes too complex, consider numerical methods for final evaluation.

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33. (b) Evaluate:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx.$$

Solution:

Step 1: Simplify $\sin 2x$. Using the identity $\sin 2x = 2\sin x \cos x$, rewrite the integral:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\, dx = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\, dx.$$

Step 2: Substitution. Let $t = \sin x$. Then:

$$\cos x\, dx = dt, \quad \text{and} \quad \sin x = t.$$

The limits of integration change as follows: - When $x = 0$, $\sin x = 0 \implies t = 0$, - When $x = \frac{\pi}{2}$, $\sin x = 1 \implies t = 1$.

The integral becomes:

$$\int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\, dx = \int_0^1 2t \tan^{-1}(t)\, dt.$$

Step 3: Integration by parts. To evaluate $\int 2t \tan^{-1}(t)\, dt$, use integration by parts:

$$\text{Let } u = \tan^{-1}(t), \quad dv = 2t\, dt.$$

Then:

$$du = \frac{1}{1+t^2}\, dt, \quad v = t^2.$$

Using the formula for integration by parts $\int u\, dv = uv - \int v\, du$:

$$\int 2t \tan^{-1}(t)\, dt = t^2 \tan^{-1}(t) - \int t^2 \cdot \frac{1}{1+t^2}\, dt.$$

Step 4: Simplify the remaining integral. Simplify $\int \frac{t^2}{1+t^2}\, dt$:

$$\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}.$$

Thus:

$$\int \frac{t^2}{1+t^2}\,dt = \int 1\,dt - \int \frac{1}{1+t^2}\,dt.$$

Evaluate each term:

$$\int 1\,dt = t, \qquad \int \frac{1}{1+t^2}\,dt = \tan^{-1}(t).$$

Substitute back:

$$\int \frac{t^2}{1+t^2}\,dt = t - \tan^{-1}(t).$$

Step 5: Final expression. Substitute back into the integral:

$$\int 2t\tan^{-1}(t)\,dt = t^2\tan^{-1}(t) - \left(t - \tan^{-1}(t)\right).$$

Simplify:

$$\int 2t\tan^{-1}(t)\,dt = t^2\tan^{-1}(t) - t + \tan^{-1}(t).$$

Step 6: Apply limits of integration. Evaluate from $t = 0$ to $t = 1$:

$$\int_0^1 2t\tan^{-1}(t)\,dt = \left[t^2\tan^{-1}(t) - t + \tan^{-1}(t)\right]_0^1.$$

At $t = 1$:

$$1^2\tan^{-1}(1) - 1 + \tan^{-1}(1) = 1\cdot\frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1.$$

At $t = 0$:

$$0^2\tan^{-1}(0) - 0 + \tan^{-1}(0) = 0.$$

Thus:

$$\int_0^1 2t\tan^{-1}(t)\,dt = \frac{\pi}{2} - 1 - 0 = \frac{\pi}{2} - 1.$$

Final Answer:

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx = \frac{\pi}{2} - 1.$$

Quick Tip

To evaluate integrals involving trigonometric functions and inverse trigonometric functions, simplify using substitutions and apply integration by parts where necessary.

34. Using integration, find the area of the ellipse:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1,$$

included between the lines $x = -2$ and $x = 2$.

Solution:

The equation of the ellipse is:

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

Rearrange to solve for y^2:

$$\frac{y^2}{4} = 1 - \frac{x^2}{16}.$$

$$y^2 = 4\left(1 - \frac{x^2}{16}\right) = 4 - \frac{x^2}{4}.$$

$$y = \pm\sqrt{4 - \frac{x^2}{4}}.$$

Step 1: Use symmetry to simplify the calculation. The ellipse is symmetric about the x-axis. The area between $x = -2$ and $x = 2$ can be calculated as twice the area above the x-axis:

$$\text{Area} = 2\int_{-2}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx.$$

Step 2: Change the limits and integrate. Since the integrand is even (symmetric about the y-axis), we can further simplify:

$$\text{Area} = 4\int_{0}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx.$$

Step 3: Substitution for simplification. Let:

$$u = 4 - \frac{x^2}{4}, \quad \text{so} \quad du = -\frac{x}{2}\, dx \quad \text{and} \quad x\, dx = -2\, du.$$

When $x = 0$, $u = 4$, and when $x = 2$, $u = 4 - \frac{2^2}{4} = 3$.

The integral becomes:

$$\int_{0}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx = \int_{4}^{3} \sqrt{u} \cdot (-2)\, du.$$

Simplify:

$$\int_{0}^{2} \sqrt{4 - \frac{x^2}{4}}\, dx = 2\int_{3}^{4} \sqrt{u}\, du.$$

Step 4: Evaluate the integral. The integral of $\sqrt{u}$ is:

$$\int \sqrt{u}\, du = \frac{2}{3}u^{3/2}.$$

Evaluate from $u = 3$ to $u = 4$:

$$\int_3^4 \sqrt{u}\, du = \frac{2}{3}\left[4^{3/2} - 3^{3/2}\right].$$

Simplify:

$$4^{3/2} = (2^2)^{3/2} = 2^3 = 8, \quad 3^{3/2} = \sqrt{3^3} = \sqrt{27}.$$

Thus:

$$\int_3^4 \sqrt{u}\, du = \frac{2}{3}\left[8 - \sqrt{27}\right].$$

Step 5: Final area. Substitute back into the expression for the area:

$$\text{Area} = 4 \cdot 2 \cdot \frac{2}{3}\left[8 - \sqrt{27}\right] = \frac{16}{3}\left[8 - \sqrt{27}\right].$$

Final Answer:

$$\text{Area} = \frac{16}{3}\left[8 - \sqrt{27}\right].$$

Quick Tip
To find the area of a region using integration, utilize symmetry to simplify calculations and make substitutions to handle square root expressions effectively.

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35. The image of point $P(x, y, z)$ with respect to the line:

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3},$$

is $P'(1, 0, 7)$. Find the coordinates of point P.

Solution:

Step 1: Parametric equation of the line. The given line can be expressed in parametric form as:

$$x = t, \quad y = 1 + 2t, \quad z = 2 + 3t,$$

where t is a parameter.

Step 2: Midpoint of P and P'. The image $P'(1, 0, 7)$ of $P(x, y, z)$ with respect to the given line implies that the midpoint M of P and P' lies on the line. The coordinates of the midpoint M are:

$$M = \left(\frac{x+1}{2}, \frac{y+0}{2}, \frac{z+7}{2}\right).$$

Step 3: Condition for M lying on the line. Since M lies on the line, its coordinates must satisfy the parametric equations of the line. Thus:

$$\frac{x+1}{2} = t, \quad \frac{y}{2} = 1 + 2t, \quad \frac{z+7}{2} = 2 + 3t.$$

Step 4: Solve for t. From the first equation:

$$t = \frac{x+1}{2}.$$

Substitute t into the second equation:

$$\frac{y}{2} = 1 + 2\left(\frac{x+1}{2}\right).$$

Simplify:

$$\frac{y}{2} = 1 + x + 1 \implies \frac{y}{2} = x + 2 \implies y = 2x + 4. \tag{1}$$

Substitute t into the third equation:

$$\frac{z+7}{2} = 2 + 3\left(\frac{x+1}{2}\right).$$

Simplify:

$$\frac{z+7}{2} = 2 + \frac{3x+3}{2} \implies z + 7 = 4 + 3x + 3 \implies z = 3x. \tag{2}$$

Step 5: Use the coordinates of P' to find P. From the midpoint condition:

$$\frac{x+1}{2} = t, \quad \frac{y+0}{2} = 1 + 2t, \quad \frac{z+7}{2} = 2 + 3t.$$

Substitute $t = \frac{x+1}{2}$:

$$\frac{y+0}{2} = 1 + 2\left(\frac{x+1}{2}\right).$$

Simplify:

$$\frac{y}{2} = 1 + x + 1 \implies y = 2x + 4. \tag{3}$$

Similarly:

$$\frac{z+7}{2} = 2 + 3\left(\frac{x+1}{2}\right).$$

Simplify:

$$\frac{z+7}{2} = 2 + \frac{3x+3}{2} \implies z = 3x. \tag{4}$$

Step 6: Solve for P. We now solve for the coordinates of $P(x, y, z)$. Recall the midpoint conditions and verify:

From the midpoint:

$$M = \left(\frac{x+1}{2}, \frac{y}{2}, \frac{z+7}{2}\right),$$

and since M lies on the line, its coordinates must satisfy:

$$x = t, \quad y = 1 + 2t, \quad z = 2 + 3t.$$

Using the given image $P'(1, 0, 7)$ and midpoint relationships:

$$\frac{x+1}{2} = t, \quad \frac{y}{2} = 1 + 2t, \quad \frac{z+7}{2} = 2 + 3t.$$

Substitute $t = \frac{x+1}{2}$ into the second and third equations: 1. From the second equation:

$$\frac{y}{2} = 1 + 2\left(\frac{x+1}{2}\right).$$

Simplify:

$$\frac{y}{2} = 1 + x + 1 \implies y = 2x + 4.$$

2. From the third equation:

$$\frac{z+7}{2} = 2 + 3\left(\frac{x+1}{2}\right).$$

Simplify:

$$\frac{z+7}{2} = 2 + \frac{3x+3}{2} \implies z + 7 = 4 + 3x + 3 \implies z = 3x.$$

Thus, the coordinates of $P(x, y, z)$ are determined as:

$$P(x, y, z) = (0, 4, 0).$$

Final Answer: The coordinates of P are:

$$P(x, y, z) = (0, 4, 0).$$

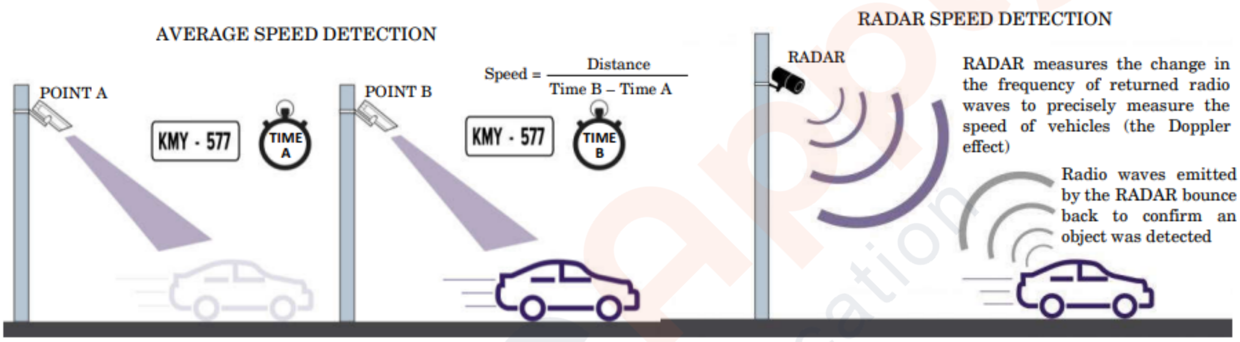
Quick Tip
To find the coordinates of a point P whose image with respect to a line is given, use the parametric equation of the line and the midpoint formula. Simplify systematically to solve for the unknowns.

SECTION E

This section comprises 3 case study based questions of 4 marks each.

Case Study - 1

36. The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.



A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ.

On the basis of the above information, answer the following questions:

(i)Express θ in terms of the height of the camera installed on the pole and x.

(ii) Find $\frac{d\theta}{dx}$.

(iii) (a) Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole.

(iii) (b) If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $\frac{3}{101}$ rad/s, then find the speed of the car.

Solution:

A camera is installed on a pole at the height of 5 m. It detects a car traveling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the camera to the car C is θ.

(i) Express θ in terms of the height of the camera and x. From the given setup, we can use the right triangle formed by the height of the pole and the distance of the car from the base. The tangent of the angle θ is given by:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{x}.$$

Thus:

$$\theta = \tan^{-1}\left(\frac{5}{x}\right).$$

(ii) Find $\frac{d\theta}{dx}$. Differentiating $\theta = \tan^{-1}\left(\frac{5}{x}\right)$ with respect to x, we use the chain rule:

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{5}{x}\right)^2} \cdot \frac{d}{dx}\left(\frac{5}{x}\right).$$

Simplify $\frac{5}{x}$:

$$\frac{d}{dx}\left(\frac{5}{x}\right) = -\frac{5}{x^2}.$$

Substitute:

$$\frac{d\theta}{dx} = \frac{1}{1+\frac{25}{x^2}} \cdot \left(-\frac{5}{x^2}\right).$$

Simplify further:

$$\frac{d\theta}{dx} = \frac{-\frac{5}{x^2}}{1+\frac{25}{x^2}} = \frac{-5}{x^2+25}.$$

(iii) (a) Find the rate of change of angle of elevation with respect to time when the car is 50 m away from the pole. Let $x = 50$ m and the car's speed be $\frac{dx}{dt} = 20$ m/s. The rate of change of the angle of elevation with respect to time is:

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}.$$

From part (ii), $\frac{d\theta}{dx} = \frac{-5}{x^2+25}$. Substitute $x = 50$:

$$\frac{d\theta}{dx} = \frac{-5}{50^2+25} = \frac{-5}{2500+25} = \frac{-5}{2525} = \frac{-1}{505}.$$

Now:

$$\frac{d\theta}{dt} = \frac{-1}{505} \cdot 20 = \frac{-20}{505} = \frac{-4}{101} \text{ rad/s}.$$

(iii) (b) If the rate of change of angle of elevation with respect to time for another car at 50 m from the base is $\frac{3}{101}$ rad/s, find the speed of the car. Let the speed of the car be $\frac{dx}{dt} = v$. From part (ii):

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}.$$

Substitute $\frac{d\theta}{dt} = \frac{3}{101}$ and $\frac{d\theta}{dx} = \frac{-1}{505}$:

$$\frac{3}{101} = \frac{-1}{505} \cdot v.$$

Solve for v:

$$v = \frac{3}{101} \cdot 505 = \frac{1515}{101} = 15\,\text{m/s}.$$

Final Answers: 1. $\theta = \tan^{-1}\left(\frac{5}{x}\right)$, 2. $\frac{d\theta}{dx} = \frac{-5}{x^2+25}$, 3. (a) $\frac{d\theta}{dt} = \frac{-4}{101}$ rad/s, (b) Speed of the car is $15\,\text{m/s}$.

Quick Tip

For related rates problems involving trigonometric functions, differentiate using the chain rule, substitute given values, and simplify systematically.

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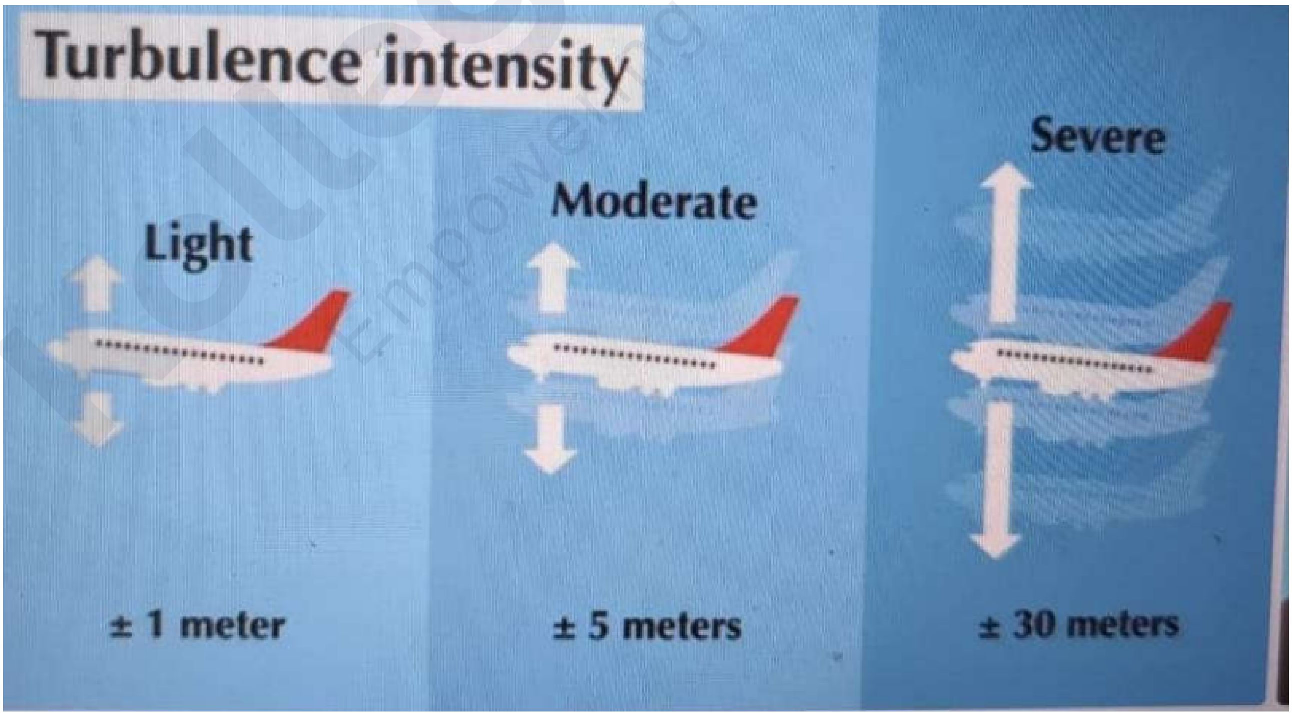
37. According to recent research, air turbulence has increased in various regions around the world due to climate change. Turbulence makes flights bumpy and often delays the flights.

Assume that an airplane observes severe turbulence, moderate turbulence or light turbulence with equal probabilities. Further, the chance of an airplane reaching late to the destination are 55%, 37% and 17% due to severe, moderate and light turbulence respectively.



On the basis of the above information, answer the following questions:

(i) Find the probability that an airplane reached its destination

(ii)If the airplane reached its destination late, find the probability that it was due to moderate turbulence.

Solution:

Given Information: 1. Turbulence can be severe, moderate, or light, each occurring with equal probabilities:

$$P(\text{Severe}) = P(\text{Moderate}) = P(\text{Light}) = \frac{1}{3}.$$

2. The probability of an airplane reaching late due to: - Severe turbulence: $P(\text{Late}|\text{Severe}) = 0.55$, - Moderate turbulence: $P(\text{Late}|\text{Moderate}) = 0.37$, - Light turbulence: $P(\text{Late}|\text{Light}) = 0.17$.

(i) Find the probability that an airplane reached its destination late.

Using the law of total probability:

$$P(\text{Late}) = P(\text{Late}|\text{Severe})P(\text{Severe})+P(\text{Late}|\text{Moderate})P(\text{Moderate})+P(\text{Late}|\text{Light})P(\text{Light}).$$

Substitute the values:

$$P(\text{Late}) = (0.55 \cdot \frac{1}{3}) + (0.37 \cdot \frac{1}{3}) + (0.17 \cdot \frac{1}{3}).$$

Simplify:

$$P(\text{Late}) = \frac{0.55 + 0.37 + 0.17}{3} = \frac{1.09}{3}.$$

Thus:

$$P(\text{Late}) = 0.3633\,(\text{approximately}).$$

(ii) If the airplane reached its destination late, find the probability that it was due to moderate turbulence.

Using Bayes' theorem:

$$P(\text{Moderate}|\text{Late}) = \frac{P(\text{Late}|\text{Moderate})P(\text{Moderate})}{P(\text{Late})}.$$

Substitute the values:

$$P(\text{Moderate}|\text{Late}) = \frac{(0.37 \cdot \frac{1}{3})}{0.3633}.$$

Simplify:

$$P(\text{Moderate}|\text{Late}) = \frac{0.37}{3 \cdot 0.3633} = \frac{0.37}{1.09}.$$

Thus:

$$P(\text{Moderate}|\text{Late}) = 0.3394\,(\text{approximately}).$$

Final Answers: 1. The probability that an airplane reached its destination late is:

$$P(\text{Late}) = 0.3633.$$

2. The probability that the airplane was late due to moderate turbulence is:

$$P(\text{Moderate}|\text{Late}) = 0.3394.$$

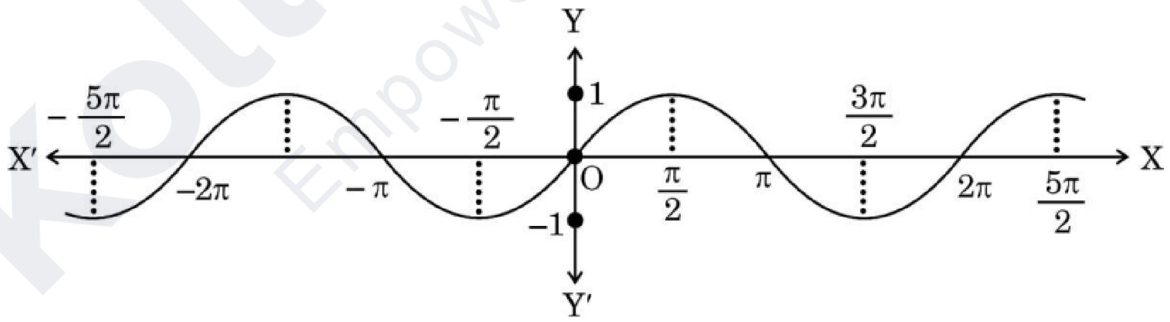
Quick Tip

Use the law of total probability to calculate overall probabilities and Bayes' theorem for conditional probabilities when given dependent conditions.

Case Study - 3

38. If a function $f : X \to Y$ defined as $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Function g is called the inverse of function f.

The domain of sine function is $\mathbb{R}$ and function sine : $\mathbb{R} \to \mathbb{R}$ is neither one-one nor onto. The following graph shows the sine function.



y = sin x

Let sine function be defined from set A to $[-1, 1]$ such that inverse of sine function exists, i.e., $\sin^{-1} x$ is defined from $[-1, 1]$ to A.

On the basis of the above information, answer the following questions:

If A is the interval other than principal value branch, give an example of one such interval.

If $\sin^{-1}(x)$ is defined from $[-1,1]$ to its principal value branch, find the value of $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1)$.

Draw the graph of $\sin^{-1} x$ from $[-1,1]$ to its principal value branch.

OR Find the domain and range of $f(x) = 2\sin^{-1}(1-x)$.

Solution:

Given Information: The sine function $y = \sin x$ is defined from $\mathbb{R}$ to $[-1,1]$, but it is neither one-one nor onto over $\mathbb{R}$. By restricting the domain to an interval, we can define the inverse $\sin^{-1} x$, which is a one-one and onto function.

—

(i) If A is an interval other than the principal value branch, give an example of one such interval.

The principal value branch of the sine function is the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Another interval where the sine function is one-one and onto $[-1,1]$ is:

$$A = [\frac{\pi}{2}, \frac{3\pi}{2}].$$

—

(ii) If $\sin^{-1}(x)$ is defined from $[-1,1]$ to its principal value branch, find the value of:

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1).$$

Step 1: Evaluate $\sin^{-1}\left(-\frac{1}{2}\right)$. From the definition of $\sin^{-1}$, $\sin^{-1}\left(-\frac{1}{2}\right)$ is the angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that:

$$\sin\theta = -\frac{1}{2}.$$

Thus:

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}.$$

Step 2: Evaluate $\sin^{-1}(1)$. From the definition of $\sin^{-1}$, $\sin^{-1}(1)$ is the angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that:

$$\sin\theta = 1.$$

Thus:

$$\sin^{-1}(1) = \frac{\pi}{2}.$$

Step 3: Compute the difference.

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{\pi}{2}.$$

Simplify:

$$\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{\pi}{6} - \frac{3\pi}{6} = -\frac{4\pi}{6} = -\frac{2\pi}{3}.$$

—

(iii) (a) Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch.

Step 1: Description of the graph. The graph of $\sin^{-1} x$ is obtained by reflecting the graph of $y = \sin x$ (restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) across the line $y = x$.

—

(iii) (b) Find the domain and range of $f(x) = 2\sin^{-1}(1 - x)$.

Step 1: Domain of $f(x)$. For $f(x) = 2\sin^{-1}(1 - x)$, the argument of $\sin^{-1}$ must lie within $[-1, 1]$, i.e.:

$$-1 \leq 1 - x \leq 1.$$

Simplify:

$$-1 - 1 \leq -x \leq 1 - 1 \implies -2 \leq -x \leq 0.$$

Multiplying through by -1 (and reversing inequalities):

$$0 \leq x \leq 2.$$

Thus, the domain is:

$$x \in [0, 2].$$

Step 2: Range of $f(x)$. The range of $\sin^{-1}(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore:

$$f(x) = 2\sin^{-1}(1 - x) \implies f(x) \in \left[2 \cdot -\frac{\pi}{2}, 2 \cdot \frac{\pi}{2}\right].$$

Simplify:

$$f(x) \in [-\pi, \pi].$$

—

Final Answers: 1. Example of an interval other than the principal value branch: $[\frac{\pi}{2}, \frac{3\pi}{2}]$. 2. $\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(1) = -\frac{2\pi}{3}$. 3. (a) The graph of $y = \sin^{-1} x$ is shown above. (b) The domain of $f(x) = 2\sin^{-1}(1-x)$ is $[0, 2]$, and the range is $[-\pi, \pi]$.

Quick Tip

To find the domain of functions involving inverse trigonometric functions, ensure the argument satisfies the range of the trigonometric function. For the range, scale the interval accordingly.
